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Instability of an elliptic jet

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The dispersion equation for waves on an infinite uniform jet column of elliptic cross-section is derived, and approximated for large eccentricity with the aid of new asymptotics for the modified Mathieu functions. It is shown that the effect of appreciable eccentricity on lateral disturbances is greatly to reduce their growth rates below those for a circular jet, regardless of whether the disturbance grows spatially or temporally. For 'vertical' disturbances it is shown that the behaviour of waves of general length is qualitatively similar to that of long waves on a two-dimensional jet. Thus the mode symmetric about the major axis has small growth rate whether the mode grows temporally or spatially, while the mode antisymmetric about the major axis has small growth rate if temporally growing, but large growth rate if spatially growing. Comments are made as to the relevance of these results to the mode of action of jet silencers which squash a round jet into a flat 'fish-tail' shape.

1. Introduction

The belief that high Reynolds number jet turbulence has a large-scale organized structure has recently been given impressively detailed and convincing support from the experimental work of Crow & Champagne (1971) and Lau, Fuchs & Fisher (1970). Equally convincing is Michalke's (1971) demonstration that an analytical model giving excellent agreement with experiment is obtained by regarding the large-scale structure as a set of *spatially growing* instabilities on the mean jet profile (notwithstanding assurances in Crow & Champagne (1971) that spatial instability is incapable of explaining the observed results). Accepting this evidence, and accepting further the (plausible, though as yet unproven) idea that jet noise and the large-scale structure are intimately related, it follows that a good case can be made for the relevance of stability calculations to the jet noise problem. Much work has, of course, already been done on those lines, particularly in the Soviet Union; see, for example, the collection of papers edited by Rimsky-Korsakov (1967) and the book by Sedel'nikov (1971), in which dispersion equations are derived for disturbances to a variety of jet configurations, including multi-layered jets, systems of jets and ejector systems.

This note gives a corresponding treatment of the jet of elliptic cross-section, motivated by the belief that it might lead to some understanding of the mode of action of jet noise supressors which squash a round jet into a flat fish-tail shape (see Voce & Simson 1972). Notches or slots may be cut into the nozzle to achieve this effect, or large plates may be used to perform the squashing, as is essentially the case in the Concorde airliner. It seems to have been generally thought that the squashed jet was quieter (in the 'flat' plane as it were, at any rate) because it had suffered better mixing, with a reduced length of potential core. If that were the case one might suppose that there would be *some* noise suppression in the other plane, and that the suppression would improve continuously with further squashing of the jet. Neither of these effects is, in fact, found. The suppression is negligible in the 'vertical' plane (containing the minor axes of the cross-sections of the jet), while even in the flat plane, the suppression increases steadily with increases in the aspect ratio of sections of the jet only up to a certain value, beyond which no further suppression results – despite the fact that the jet continues to spread ever more rapidly. It appeared that a more likely explanation of the quietness of the squashed jet in the flat plane might come from a study of the instability characteristics of the jet, and that is the objective here. Expediency requires that the simplest model of the squashed jet be adopted, in which the flow is taken to be uniform and incompressible within a cylindrical vortex sheet whose cross-section is an ellipse. Real velocity profiles do not, of course, have this 'top-hat' form, except perhaps very close to the nozzle (where we could not, in any case, ignore the presence of the nozzle), and Michalke (1971) has shown that it is important to take a fairly realistic profile if good quantitative agreement with experiment is to be obtained. Such agreement is not the aim here, however, and it is unlikely that the 'top-hat' profile is so unrealistic as to disguise essential features. A more serious error here probably lies in the adoption of a profile which is unchanging with axial distance, especially as the squashed jet does spread more rapidly than the round jet. But in any case, a satisfactory way of incorporating appreciable spreading even into stability theory for a plane shear layer is not yet in sight, so that we do not consider that aspect further.

2. The eigenvalue equation

We consider uniform incompressible flow at speed U within the elliptical vortex sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \quad (-\infty < z < +\infty).$$

Outside the vortex sheet the fluid is stagnant. We impose (irrotational) disturbances on the flow, with an implied axial and time dependence $\exp[i\alpha z - i\omega t]$ throughout. For the moment we restrict ourselves to *temporal* instability, so that the complex frequency ω is to be found in terms of the real positive wavenumber α.

Denoting the perturbation potentials outside and inside the jet by $\phi(x,y)$ and $\psi(x,y)$ respectively, we have to satisfy the Helmholtz equations

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\alpha^2\right)\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 0 \tag{2.1}$$

with the usual kinematic and dynamic conditions

$$\left.\begin{aligned}(\omega-\alpha U)\frac{\partial\phi}{\partial n} &= \omega\frac{\partial\psi}{\partial n},\\ \omega\phi &= (\omega-\alpha U)\psi\end{aligned}\right\} \tag{2.2}$$

on the ellipse, $\partial/\partial n$ denoting differentiation along the normal. In addition, ϕ must vanish at infinity, while ψ and $\nabla\psi$ must be finite and continuous everywhere within the ellipse.

Introduce elliptic cylinder co-ordinates according to

$$x+iy = a\epsilon\cosh(\rho+i\theta),$$

where ϵ is the eccentricity, $\rho \geqslant 0$ and $0 \leqslant \theta \leqslant 2\pi$. Then problem (2.1)–(2.2) becomes

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial\theta^2}-\frac{\alpha^2a^2\epsilon^2}{2}(\cosh 2\rho-\cos 2\theta)\right]\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 0, \tag{2.3}$$

$$\left.\begin{aligned}\frac{\partial\psi}{\partial\rho}(\rho_0,\theta) &= \left(1-\frac{U\alpha}{\omega}\right)\frac{\partial\phi}{\partial\rho}(\rho_0,\theta),\\ \phi(\rho_0,\theta) &= \left(1-\frac{U\alpha}{\omega}\right)\frac{\partial\psi}{\partial\rho}(\rho_0,\theta),\end{aligned}\right\} \tag{2.4}$$

$\rho = \rho_0$ defining the ellipse. A separable solution of (2.3) exists in the form

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \mathscr{P}(\rho)\,\Theta(\theta)$$

provided

$$\Theta''+(\lambda+2q\cos 2\theta)\,\Theta = 0, \tag{2.5}$$

$$\mathscr{P}''-(\lambda+2q\cosh 2\rho)\mathscr{P} = 0, \tag{2.6}$$

with $q = (\frac{1}{2}\alpha a\epsilon)^2$, these being, respectively, the Mathieu equation and modified Mathieu equation, with parameter $-q$.

Solutions periodic in θ with period π or 2π, such that ψ and $\nabla\psi$ are continuous for $\rho < \rho_0$ (in particular, across the interfocal line) and such that $\phi \to 0$ as $\rho \to \infty$, are (see for example, McLachlan 1947, p. 294 *et seq.*)

$$\left.\begin{aligned}\psi &= A\,\mathrm{Ce}_m(\rho,-q)\,\mathrm{ce}_m(\theta,-q),\\ \phi &= B\,\mathrm{Fek}_m(\rho,-q)\,\mathrm{ce}_m(\theta,-q),\end{aligned}\right\} \tag{2.7}$$

in which λ must have a characteristic value generally denoted by $a_{2n}(q)$ when $m = 2n$, while $\lambda = b_{2n+1}(q)$ when $m = 2n+1$. These solutions are even about the major axis, and are even or odd about the minor axis according as $m = 2n$ or $m = 2n+1$. Solutions odd about the major axis, and even or odd about the minor axis according as $m = 2n+1$ or $m = 2n+2$, are

$$\left.\begin{aligned}\psi &= A\,\mathrm{Se}_m(\rho,-q)\,\mathrm{se}_m(\theta,-q),\\ \phi &= B\,\mathrm{Gek}_m(\rho,-q)\,\mathrm{se}_m(\theta,-q),\end{aligned}\right\} \tag{2.8}$$

where $\lambda = a_{2n+1}(q)$ for $m = 2n+1$ and $\lambda = b_{2n+2}(q)$ for $m = 2n+2$. In all these relations, $n = 0, 1, 2, \ldots$, and n signifies the number of zeros of the angular functions between 0 and $\frac{1}{2}\pi$. The functions $\mathrm{ce}_m\,\theta$ and $\mathrm{se}_m\,\theta$ are analogous to $\cos m\theta$ and $\sin m\theta$, while $\mathrm{Ce}_m\,\rho$ and $\mathrm{Se}_m\,\rho$ are analogous to $I_m(\alpha r)$ and $\mathrm{Fek}_m\,\rho$ and $\mathrm{Gek}_m\,\rho$ to $K_m(\alpha r)$ in the case of the round jet.

Eigenvalue equations then follow directly from (2.4) as

$$\left(\frac{U\alpha}{\omega}-1\right)^2 = \frac{\mathrm{Ce}'_m(\rho_0,-q)\,\mathrm{Fek}_m(\rho_0,-q)}{\mathrm{Ce}_m(\rho_0,-q)\,\mathrm{Fek}'_m(\rho_0,-q)}, \tag{2.9}$$

$$\left(\frac{U\alpha}{\omega}-1\right)^2=\frac{\mathrm{Se}'_m(\rho_0,-q)\,\mathrm{Gek}_m(\rho_0,-q)}{\mathrm{Se}_m(\rho_0,-q)\,\mathrm{Gek}'_m(\rho_0,-q)}, \tag{2.10}$$

in which we shall abbreviate the right sides in an obvious way as $W_m(C,F)$ or $W_m(S,G)$. These expressions are the direct analogues of the equation

$$\left(\frac{U\alpha}{\omega}-1\right)^2=\frac{I'_m(\alpha a)\,K_m(\alpha a)}{I_m(\alpha a)\,K'_m(\alpha a)}$$

given by Batchelor & Gill (1962) for mth-order azimuthal disturbances to a circular jet of radius a. For that problem it is easy to show that $W_m(I,K)$ is real and negative for all m and all αa, so that all disturbances are unstable. In the present problem we have found no such general result, though neither have we found any case of purely neutral stability, so that it seems unlikely that the ellipticity alone can completely stabilize the flow to any class of disturbances. None the less, some interesting results are found when the eccentricity is large, and suitable approximations for that case are found in the next section.

3. Approximations for large eccentricity

Let $R=(ab)^{\frac{1}{2}}$ be the radius of the equivalent-area circle, and consider the limit $\epsilon\to 1-$ with $\alpha R=O(1)$. Then

$$\rho_0\sim[2(1-\epsilon)]^{\frac{1}{2}},\quad q\sim(\tfrac{1}{2}\alpha R)^2[2(1-\epsilon)]^{-\frac{1}{2}}, \tag{3.1}$$

and we therefore need asymptotics for the modified Mathieu functions, as $q\to\pm\infty$, holding uniformly in ρ down to values as small as q^{-1}.

Consider for definiteness the case $\lambda=a_{2n}(q)$, for which (2.6) has solutions $\mathrm{Ce}_{2n}(\rho,-q)$ and $\mathrm{Fek}_{2n}(\rho,-q)$. As $q\to+\infty$ we have (McLachlan 1947, p. 239)

$$a_{2n}(q)=-2q+(8n+2)\,q^{\frac{1}{2}}+O(1), \tag{3.2}$$

where the $O(1)$ and smaller terms are known, but not needed here. The equation for $\mathscr{P}(\rho)$ is then

$$\mathscr{P}''-\{4q\sinh^2\rho+(8n+2)\,q^{\frac{1}{2}}+O(1)\}\mathscr{P}=0, \tag{3.3}$$

and a straightforward WKB expansion

$$\mathscr{P}(\rho)\sim\exp\,[q^{\frac{1}{2}}f(\rho)]\,[g_0(\rho)+q^{-\frac{1}{2}}g_1(\rho)+\dots]$$

gives

$$\left.\begin{aligned}f(\rho)&=\pm 2\cosh\rho,\\ g_0(\rho)&=(\operatorname{cosech}\rho)^{\frac{1}{2}}\,(\tanh\tfrac{1}{2}\rho)^{\pm(2n+\frac{1}{2})},\end{aligned}\right\} \tag{3.4}$$

with the same choice of $\pm$. These approximations are proportional to $\mathrm{Ce}_{2n}(\rho,-q)$ (plus sign) and to $\mathrm{Fek}_{2n}(\rho,-q)$ (minus sign). A plausible way of seeing this is to let $\rho\to+\infty$, in which case the $\mathscr{P}_{\pm}(\rho)$ behave like

$$r^{-\frac{1}{2}}\exp\,[\pm\alpha r]\quad(r=(x^2+y^2)^{\frac{1}{2}}),$$

which are proportional to the asymptotic forms of Ce_{2n} and Fek_{2n} respectively (McLachlan 1947, p. 369). A rigorous procedure involves using the WKB series to get approximations to the ordinary periodic Mathieu functions, identifying them by parity and periodicity considerations, and then using various transformations to get to the above results for the modified Mathieu functions.

If further terms in the WKB expression are now pursued we find

$$g_1/g_0 = O(\cosh\rho/q^{\frac{1}{2}}\sinh^2\rho).$$

In particular, for ρ as small as q^{-1} successive terms increase by $O(q^{\frac{3}{2}})$ and the series is not uniformly valid in ρ. Here we get round the difficulty by finding a complementary expansion for small values of ρ and matching it to the solution (3.4). Since the region of non-uniformity is essentially $\rho = 0$ to $q^{-\frac{1}{4}}$ we write $\tau = 2\rho q^{\frac{1}{4}}$ and look for an asymptotic solution of (3.3) as $q \to +\infty$ with $\tau = O(1)$.

To leading order we then have

$$(d^2\mathscr{P}/d\tau^2) - (\tfrac{1}{4}\tau^2 + 2n + \tfrac{1}{2})\,\mathscr{P} = 0, \tag{3.5}$$

whose solutions are parabolic cylinder functions. These functions have been used to approximate Mathieu functions before (McLachlan 1947, p. 369; Abramowitz & Stegun 1964, p. 742), but all those approximations relate solely to ordinary Mathieu functions and cannot be used to find the values of the *modified* functions near $\rho = 0$. Consider first the solution corresponding to Ce_{2n}. We need a solution capable of matching the function $\mathscr{P}_+(\rho)$ of (3.4), and it is easily seen that this requires the corresponding solution of (3.5) to behave like $\tau^{2n}\exp(\frac{1}{4}\tau^2)$ as $\tau \to +\infty$. Further, the solution must be even in τ, since $\mathrm{Ce}_{2n}\,\rho$ is even in ρ. These conditions determine the solution uniquely up to a multiplicative constant (depending upon q) as

$$\mathrm{Ce}_{2n}(\rho, -q) \sim \mathscr{P}_+(\tau) = D_{-2n-1}(\tau) + D_{-2n-1}(-\tau). \tag{3.6}$$

For the solution of (3.5) representing $\mathrm{Fek}_{2n}\rho$ we have no parity requirement. This time the condition that the solution match the $\mathscr{P}_-(\rho)$ of (3.4) requires a behaviour $\tau^{-(2n+1)}\exp[-\frac{1}{4}\tau^2]$ and is sufficient to determine the solution uniquely, up to a multiplicative constant, as

$$\mathrm{Fek}_{2n}(\rho, -q) \sim \mathscr{P}_-(\tau) = D_{-2n-1}(\tau). \tag{3.7}$$

The notation here is standard for solutions of (3.5). Note that, if attention were paid to the functions of q involved in the matching and in the precise identification of (3.4) with standard definitions of Ce and Fek, it would be possible to use (3.6) and (3.7) to describe in detail the behaviour of the modified Mathieu functions for large q and all small ρ, a description which appears not to have been given previously.

Turning now to other cases, the expansion (3.2) holds also for $b_{2n+1}(q)$, and the matching and parity requirements are also the same as in the above case. Thus, to the order considered here, $\mathrm{Ce}_{2n+1}(\rho, -q) = \mathrm{Ce}_{2n}(\rho, -q)$ and is given by (3.6), while $\mathrm{Fek}_{2n+1}(\rho, -q) = \mathrm{Fek}_{2n}(\rho, -q)$ and is given by (3.7). The other possibilities, $\lambda = a_{2n+1}(q)$ and $\lambda = b_{2n+2}(q)$, are both covered by (3.2) provided that $8n+2$ is replaced on the right by $8n+6$. Apart from that, and the fact that we now need an odd solution of (3.5) to represent $\mathrm{Se}_m(\rho, -q)$, the work is unchanged, and we find

$$\left.\begin{matrix}\mathrm{Se}_{2n+1}(\rho, -q)\\ \mathrm{Se}_{2n+2}(\rho, -q)\end{matrix}\right\} \sim D_{-2n-2}(\tau) - D_{-2n-2}(-\tau), \tag{3.8}$$

$$\left.\begin{matrix}\mathrm{Gek}_{2n+1}(\rho, -q)\\ \mathrm{Gek}_{2n+2}(\rho, -q)\end{matrix}\right\} \sim D_{-2n-2}(\tau). \tag{3.9}$$

Of these results, those for Ce_{2n} and Se_{2n+1} can be found from transformation of well-known results (McLachlan 1947, p. 370); the remainder do not seem to have been recorded before. The relations (3.6)–(3.9) can now be used in (2.9)–(2.10), and can be further approximated, since, according to (3.1), $\tau_0 = 2\rho_0 q^{\frac{1}{4}}$ is in fact $o(1)$ as $q \to \infty$. We then find

$$W_{2n}(C,F) = W_{2n+1}(C,F) = -\frac{(2n+\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{2^{\frac{1}{2}}\Gamma(n+1)}\tau_0, \tag{3.10}$$

$$W_{2n+1}(S,G) = W_{2n+2}(S,G) = -\frac{\Gamma(n+1)}{2^{\frac{1}{2}}\Gamma(n+\frac{3}{2})}\tau_0^{-1}. \tag{3.11}$$

To help visualize the kind of motion described by these equations one can regard the $W_{2n}(C,F)$ modes as analogous to the axisymmetric mode of a circular jet in that they are symmetric about both axes, while the $W_{2n+1}(C,F)$ modes are even about the major axis, odd about the minor, and so represent sideways oscillations in the plane of the major axes. The $W_{2n+1}(S,G)$ modes, on the other hand, are even about the minor axis, odd about the major, and thus represent a flapping of the jet column in the plane of the minor axes, while the $W_{2n+2}(S,G)$ modes are odd about both axes and describe a kind of torsional oscillation of the column.

Only low values of n are relevant in practice (and in any case the approximations leading to (3.10)–(3.11) are invalid when $n \gg 1$). We see then that the phase speeds of the (C,F) modes are equal to U, while the temporal amplification rates are small, of order $\tau_0^{\frac{1}{2}}$, whether $m = 2n$ or $2n+1$; and that the amplification rates of the (S,G) modes are also small and $O(\tau_0^{\frac{1}{2}})$, but that the (S,G) modes have *small* phase speeds $O(\tau_0 U)$. Note here that the wavelength-to-radius parameter αR has been held at a general value $O(1)$. Thus in all cases, and for a general $\alpha R = O(1)$, the jet of large eccentricity is virtually stable to temporally growing modes, and additionally can only support (S,G) modes of very low phase speed.

These are also the qualitative features of the two-dimensional jet *in the long wavelength limit*. There, if the jet width is $2B$, we have

$$[(U\alpha/\omega)-1]^2 = -\tanh\alpha B \sim -\alpha B \tag{3.12}$$

for the symmetric mode (analogous to the Ce_0, Fek_0 mode), while

$$[(U\alpha/\omega)-1]^2 = -\coth\alpha B \sim -(\alpha B)^{-1} \tag{3.13}$$

for the antisymmetric flapping mode (analogous to the Se_1, Gek_1 mode of the elliptic jet). Writing $\tau_1 = \alpha B$, the long wave properties of the two-dimensional jet are essentially those of the elliptic jet of large eccentricity with a general αR, τ_1 corresponding to τ_0.

Although the behaviour of the (C,F) modes seem reasonable enough, that of the (S,G) modes – in particular, their *small* growth rate – seems at variance with intuition. Rather than a small growth rate, one would expect the flapping kinds of mode of an elliptic jet to have *large* growth rates. The difference arises because the above results relate to temporal instability, while intuition is more naturally based on ideas of spatial growth, at any rate for jet flows. Assume that the results for the elliptic jet hold generally for complex α and ω (a proof would require the justification of a number of steps in the foregoing, in particular that (3.2) continues to hold for complex q). Then we can find the spatial instability characteristics (ω real, α complex) for ϵ close to unity and for an arbitrary but $O(1)$ Strouhal number $S = \omega R/U$. For the (C, F) modes we find

$$(U\alpha/\omega) - 1 \sim \pm iS^{\frac{1}{4}}(1-\epsilon)^{\frac{3}{16}}, \tag{3.14}$$

while for the (S, G) modes

$$(U\alpha/\omega) - 1 \sim \exp\left(\pm \tfrac{2}{5}\pi i\right) S^{-\frac{1}{5}}(1-\epsilon)^{-\frac{3}{20}}, \tag{3.15}$$

showing, as anticipated, that the (C, F) modes have small growth rates, while the (S, G) modes have large growth rates, these trends being accentuated if $S \ll 1$. Again, these features are qualitatively the same as those of the two-dimensional jet at *low* values of S, as can be seen by inverting (3.12) and (3.13) to give

$$(U\alpha/\omega) - 1 \sim \pm iS^{\frac{1}{2}} \quad \text{(symmetric mode)}, \tag{3.16}$$

$$(U\alpha/\omega) - 1 \sim \exp\left(\pm \tfrac{1}{3}\pi i\right) S^{-\frac{1}{3}} \quad \text{(antisymmetric mode)}, \tag{3.17}$$

with $S = \omega B/U \ll 1$.

4. Conclusions

It has been shown that an elliptic-section jet of large eccentricity is virtually stable to all temporally growing disturbances, and that, in the case of spatially growing disturbances, modes representing sideways oscillation parallel to the major axis have small growth rate, while those representing a flapping motion parallel to the minor axis have large growth rate. These features go some way towards supporting the idea that certain kinds of jet nozzle achieve noise suppression in particular planes by modifying the mean flow stability characteristics with respect to disturbances in those planes. Correspondingly, one is encouraged to look for trends of a similar kind emerging from other stability calculations, in the hope of finding other potential means for noise suppression. In this light it is perhaps interesting to remark that the noise field of two parallel round jets can under some conditions greatly exceed that of one jet, and that the explanation may lie in the introduction of some unfortunate instability characteristics through mutual interaction effects. Sedel'nikov (1971, p. 14 *et seq.*) has obtained the dispersion equation for two parallel round jets, but has not developed the study in sufficient detail to see whether, in fact, this is the case.

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